

C-INDEPENDENCE AND C-RANK OF POSETS AND LATTICES

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ABSTRACT. Continuing with the authors concept (and results) of defining independence for columns of a boolean and superboolean matrix, we apply this theory to finite lattices and finite posets, introducing boolean and superboolean matrix representations for these objects. These representations yield the new concept of c -independent subsets of lattices and posets, for which the notion of c -rank is determined as the cardinality of the largest c -independent subset. We characterize this c -rank and show that c -independent subsets have a very natural interpretation in term of the maximal chains of the Hasse diagram and the associated partitions of the lattice. This realization has direct important connections with chamber systems.

INTRODUCTION

The concept of boolean and superboolean representations had been introduced first in [6] for finite hereditary collections, and later was studied in depth for matroids [7]. In the present paper we broaden this concept to finite partially ordered sets (written posets, as usual) and mainly to finite lattices. Furthermore, we show that the same representation ideas are naturally applicable to structured sets, either finite or infinite (cf. §2).

Our representations are performed by using matrices with coefficients over the superboolean semiring [6], a certain instance of a finite supertropical semiring [4, 8]. The algebra of these matrixes provides a proper notion of linear independence [5], without the use of negation, which is absent in the “weak” structure of semirings. This notion of independence, determined for lattices and posets via their representations, is at the heart of our theory and leads neutrally to the introduction of the c -rank, defined to be the cardinality of the largest independent subset.

When dealing with lattices, independent subsets have a fundamental correspondence with the maximal chains of the lattice. In particular, we prove that the c -rank of a finite lattice equals its height (Theorem 4.5). Introducing the idea of “pushing chains” (cf. Definition 4.7), we show that this pushing operation preserves lattice independent subsets.

The perspective of the Hasse diagram, together with that of Dedekind-MacNeille completion, leads us to partition of lattices – a novel idea – which plays a major role in our theory. Another important notion in our theory is that of “partial cross sections”, which provides a characterization of properties of a subset with respect to a certain partition. All these enable us to determine the fundamental connection between independence of lattice subsets (determined by its matrix representation) and the actual lattice structure (Theorem 4.15).

Finally, assisted by boolean modules and their corresponding lattices, we apply our representation techniques to finite hereditary collections (also known as finite abstract simplicial complexes), yielding additional connections between the boolean representation of these objects and finite lattices (Theorems 5.4 and 5.5).

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Notation. In this paper, for simplicity, we use the following notation: Given a subset $X \subseteq E$, and elements $x \in X$ and $p \in E$, we write $X - x$ and $X + y$ for $X \setminus \{x\}$ and $X \cup \{y\}$, respectively; accordingly we write $X - x + y$ for $(X \setminus \{x\}) \cup \{y\}$.

1. BOOLEAN AND SUPERBOOLEAN ALGEBRA

The very well known **boolean semiring** is the two element idempotent semiring $\mathbb{B} := (\{0, 1\}, +, \cdot)$, whose addition and multiplication are given respectively by the following tables:

$$\begin{array}{c|cc} + & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 1 \end{array} \quad \text{and} \quad \begin{array}{c|cc} \cdot & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 1 \end{array}.$$

The **superboolean semiring** $\mathbb{SB} := (\{1, 0, 1^\nu\}, +, \cdot)$ is three element supertropical semiring [8], a “cover” of the boolean semiring, endowed with the two binary operations:

$$\begin{array}{c|ccc} + & 0 & 1 & 1^\nu \\ \hline 0 & 0 & 1 & 1^\nu \\ 1 & 1 & 1^\nu & 1^\nu \\ 1^\nu & 1^\nu & 1^\nu & 1^\nu \end{array} \quad \begin{array}{c|ccc} \cdot & 0 & 1 & 1^\nu \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1^\nu \\ 1^\nu & 0 & 1^\nu & 1^\nu \end{array}$$

addition and multiplication, respectively. This semiring is totally ordered by $1^\nu > 1 > 0$. Note that \mathbb{SB} is **not** an idempotent semiring, since $1 + 1 = 1^\nu$, and thus \mathbb{B} is **not** a subsemiring of \mathbb{SB} . The element 1^ν is called the **ghost** element, where $\mathcal{G}_0 := \{0, 1^\nu\}$ is the **ghost ideal**¹ of \mathbb{SB} .

1.1. Boolean matrices. The semiring $M_n(\mathbb{SB})$ of $n \times n$ superboolean matrices with entries in \mathbb{SB} is defined in the standard way, where addition and multiplication are induced from the operations of \mathbb{SB} as in the familiar matrix construction. The unit element I of $M_n(\mathbb{SB})$, is the matrix with 1 on the main diagonal and whose off-diagonal entries are all 0.

A typical matrix is often denoted as $A = (a_{i,j})$, and the zero matrix is written as (0) . A matrix is said to be a **ghost matrix** if all of its entries are in \mathcal{G}_0 . A boolean matrix is a matrix with coefficients in $\{0, 1\}$, the subset of boolean matrices is denoted by $M_n(\mathbb{B})$.

The following discussion is presented for superboolean matrices, where boolean matrices are considered as superboolean matrices with entries in $\{0, 1\}$. Note that boolean matrices $M_n(\mathbb{B})$ are **not** a sub-semiring of the semiring of superboolean matrices $M_n(\mathbb{SB})$.

In the standard way, for any matrix $A \in M_n(\mathbb{SB})$, we define the **permanent** of $A = (a_{i,j})$ as:

$$\text{per}(A) := \sum_{\pi \in S_n} a_{\pi(1),1} \cdots a_{\pi(n),n} \quad (1.1)$$

where S_n stands for the group of permutations of $\{1, \dots, n\}$. Note that the permanent of a boolean matrix can be 1^ν . We say that a matrix A is **nonsingular** if $\text{per}(A) = 1$, otherwise A is said to be **singular**.

Lemma 1.1 ([6, Lemma 3.2]). *A matrix $A \in M_n(\mathbb{SB})$ is nonsingular iff by independently permuting columns and rows it has the triangular form*

$$A' := \begin{pmatrix} 1 & 0 & \cdots & 0 \\ * & \ddots & \ddots & \vdots \\ \vdots & \ddots & 1 & 0 \\ * & \cdots & * & 1 \end{pmatrix}, \quad (1.2)$$

with all diagonal entries 1, all entries above the diagonal are 0, and the entries below the diagonal belong to $\{1, 1^\nu, 0\}$.

Such reordering of A is equivalent to multiplying the matrix A by two permutation matrices Π_1 and Π_2 on the right and on the left, respectively, i.e., $A' := \Pi_1 A \Pi_2$.

¹In the supertropical setting, the elements of the complement of \mathcal{G}_0 are called **tangibles**.

Let A be an $m \times n$ superboolean matrix. We say that a $k \times \ell$ matrix B , with $k \leq m$ and $\ell \leq n$, is a **submatrix** of A if B can be obtained by deleting rows and columns of A . In particular, a **row** of a matrix A is an $1 \times n$ submatrix of A , where a **subrow** of A is an $1 \times \ell$ submatrix of A , with $\ell \leq n$. A **minor** is a submatrix obtained by deleting exactly one row and one column of a square matrix.

Definition 1.2 ([6, Definition 3.3]). A **marker** ρ in a matrix is a subrow having a single 1-entry and all whose other entries are 0; the length of ρ is the number of its entries. A marker of length k is written k -marker.

For example the nonsingular matrix A' in (1.2) has a k -marker for each $k = 1, \dots, n$, appearing in this order from bottom to top. (Note that in general markers need not be disjoint.)

Corollary 1.3 ([6, Corollary 3.4]). If a matrix $A \in M_n(\mathbb{SB})$ is a nonsingular matrix, then A has an n -marker.

Definition 1.4 ([5, Definition 1.2]). A collection of vectors $v_1, \dots, v_m \in \mathbb{SB}^{(n)}$ is said to be **dependent** if there exist $\alpha_1, \dots, \alpha_m \in \{0, 1\}$, not all of them 0, for which

$$\alpha_1 v_1 + \dots + \alpha_m v_m \in \mathcal{G}_0^{(n)}.$$

Otherwise the vectors are said to be **independent**.

The **column rank** of a superboolean matrix A is defined to be the maximal number of independent columns of A . The **row rank** is defined similarly with respect to the rows of A .

Theorem 1.5 ([5, Theorem 3.11]). For any supertropical matrix A the row rank and the column rank are the same, and this rank is equal to the size of the maximal nonsingular submatrix of A .

Definition 1.6. Let $A = (a_{i,j})$ be a superboolean matrix. The **complement** $A^c := (a_{i,j}^c)$ of A is defined by the rule $a_{i,j}^c = 1 \Leftrightarrow a_{i,j} = 0$, and $a_{i,j}^c = 1^\nu$ for every ghost entry $a_{i,j} = 1^\nu$. The **transpose** $A^t = (a_{i,j}^t)$ of A is given by $a_{i,j}^t = a_{j,i}$.

Then we can conclude the following:

Corollary 1.7. The rank of a superboolean matrix is invariant under

- (i) permuting of rows (columns);
- (ii) deletion of a row (column) whose entries are all in \mathcal{G}_0 ;
- (iii) deletion of a repeated row or column;
- (iv) transposition, i.e., $\text{rk}(A) = \text{rk}(A^t)$.

Proof. Immediate by Theorem 1.5. □

Proposition 1.8. Transposition and complement commute, i.e., $(A^t)^c = (A^c)^t$ for any superboolean matrix $A \in M_n(\mathbb{SB})$.

Proof. Straightforward: $(a_{i,j}^t)^c = (a_{j,i})^c = (a_{i,j}^c)^t$. □

Notation 1.9. Given a matrix A and a subset $Y \subseteq \text{Col}(A)$ of columns of A , we write $A[* , Y]$ for the submatrix of A having the columns Y . Sometimes we refer to $\text{Col}(A)$ as a collection of vectors, but no confusion should arise. Given also a subset $X \subseteq \text{Row}(A)$ of rows of A , we define $A[X, Y]$ to be the submatrix of A having the intersection of columns Y and the rows X , often also referred to as a collection of sub-vectors.

2. ABSTRACT SETTING

2.1. Structured sets. Let X be a nonempty finite set, i.e., $|X| = n$, and let \mathcal{R} be a binary relation defined on the elements of X , written $x_i \mathcal{R} x_j$; thus \mathcal{R} determines a structure on X . We denote such a pair by (X, \mathcal{R}) , and call it a **structured set** (over X).

Given a structured set (X, \mathcal{R}) , with $X := \{x_1, \dots, x_n\}$ a finite set of elements, we associate X with the $n \times n$ boolean matrix $A(X) := (a_{i,j})$, called **structure matrix**, defined as

$$a_{i,j} := \begin{cases} 1 & \text{if } x_i \mathcal{R} x_j, \\ 0 & \text{otherwise.} \end{cases} \quad (2.1)$$

We write (X, A) for the set X and together with its structure matrix $A = A(X)$ defined above, and call this pair again a **structured set**.

Having the above construction it is clear that the relation \mathcal{R} is fully recorded by the matrix A and vice versa. Therefore, we identify the relation \mathcal{R} on X with the matrix $A := A(X)$.

2.2. Independence. we open with the key definition of our further development:

Definition 2.1. *Given a structured set $X := (X, A)$ we define the **c-rank** of X as*

$$\text{c-rk}(X) := \text{rk}(A^c), \quad A^c := (A(X))^c.$$

Given a structured set $X := (X, A)$, consider the matrix A^c , written also as $A^c := A^c(X)$. We say that a subset $W \subseteq X$ is **c-independent** if the columns $A^c[* , W]$ of A^c corresponding to W are independent in the sense of Definition 1.4. When $|W| = k$, these columns contains a $k \times k$ nonsingular submatrix $A^c[U, W]$ with $U \subseteq X$ and $|U| = k$ (cf. Theorem 1.5), which we call a **witness** of W (in A^c). Abusing terminology, we also say that U is a witness of W in the set X . Permuting independently the columns of a witness, it has the triangular Form (1.2), cf. Lemma 1.1.

Accordingly, suppose $X := (X, A)$, $|X| = n$, is a structured set and let $W \subseteq X$ be an independent subset with $|W| = k$. Then we have the following properties satisfied:

- (a) $\text{c-rk}(W) = k \leq n$,
- (b) $\text{c-rk}(W) \leq \text{c-rk}(X)$,
- (c) $\text{c-rk}(X) \leq n$.

3. FINITE LATTICES AND FINITE BOOLEAN MODULES

In this section we start an explicit study of certain classes of structure sets, equipped with extra properties.

3.1. Posets. A major example for a structure set, and the most abstract in this paper, is given by the following well known definition [2, 15].

Definition 3.1. *A **partial order** is a binary relation \leq over a set X which is reflexive, antisymmetric, and transitive, i.e., for all $a, b, c \in X$, we have that:*

- (i) $a \leq a$ (reflexivity);
- (ii) if $a \leq b$ and $b \leq a$ then $a = b$ (antisymmetry);
- (iii) if $a \leq b$ and $b \leq c$ then $a \leq c$ (transitivity).

A pair (P, \leq) , with \leq a partial order, is called a **partially order set** – **poset** for short.

The **reverse poset** $P_{\text{rvs}} := (P, \geq)$ of $P := (P, \leq)$ is defined by reversing the order of P , i.e.,

$$p \leq q \text{ in } P \Leftrightarrow p \geq q \text{ in } P_{\text{rvs}}.$$

Definition 3.2. *Given a poset element $p \in P$, we define the **up-set** p^\uparrow and the **down-set** p^\downarrow of p respectively as*

$$p^\uparrow := \{x \in P \mid x \geq p\}, \quad \text{and} \quad p^\downarrow := \{x \in P \mid x \leq p\}.$$

A subset $I \subseteq P$ of a poset $P := (P, \leq)$ is an **order ideal** if for

$$p \in I \text{ and } q \leq p \Rightarrow q \in I.$$

Accordingly, for each $p \in P$, the down-set p^\downarrow is an order ideal of P .

Remark 3.3. The structure matrix $A(P) := (a_{i,j})$ of a poset $P := (P, \leq)$, cf. (2.1), is given by

$$a_{i,j} := \begin{cases} 1 & \text{if } p_i \leq p_j, \\ 0 & \text{otherwise,} \end{cases} \quad (3.1)$$

and therefore has the proprieties

- (a) $a_{i,i} = 1$, by reflexivity, for every $i = 1, \dots, n$.
- (b) $a_{i,j} = 1$ iff $a_{j,i} = 0$, by antisymmetry, for any $i \neq j$.

Proposition 3.4. The reversing P_{rvs} of a poset $P := (P, \leq)$, recorded by (P, A) , is equivalent to (P, A^t) .

Proof. Obtained immediately by Remark 3.3. \square

Corollary 3.5. Given a poset $P := (P, \leq)$, reversing the order on P does not change the rank of P , that is $\text{c-rk}(P) = \text{c-rk}(P_{\text{rvs}})$.

Proof. Clear from Proposition 3.4 and Corollary 1.7. \square

We recall an additional known non-negative function on posets (the c-rank was one of them, defined earlier in general for structured set).

Definition 3.6. The **height** of a poset $P := (P, \leq)$, written $\text{ht}(P)$, is defined to be the length of the longest strict chain it contains, i.e.,

$$\text{ht}(P) := \max\{k \mid p_0 < p_1 < \dots < p_k, \quad p_0, p_1, \dots, p_k \in P\}.$$

3.2. Semilattices. Let us recall some standard definitions [2, 15].

Definition 3.7. A (finite) poset $P := (P, \leq)$ whose elements admit a **join** relation (also known as the least upper bound, or the supremum), i.e., $p_i \vee p_j$ for all $p_i, p_j \in P$, is called a (finite, join) **semilattice**, denoted as $S := (S, \leq)$.

We define the category SLAT of semilattices, whose maps are sup-maps, given as follows ($\bigvee X$ stands for the common join of the members of X):

Definition 3.8. A **semilattice map**

$$\varphi : (S, \leq) \longrightarrow (S', \leq)$$

that satisfies $\varphi(\bigvee X) = \bigvee(\varphi(X))$ for all $X \subseteq S$ is called a **sup-map**.

Any semilattice (S, \leq) can be viewed as a semigroup $(S, +)$ by defining

$$s + t := t \vee s, \quad \text{for any } s, t \in S.$$

This semigroup is commutative ($s + t = t + s$) and idempotent ($s + s = s$ for any $s \in S$). When a (join) semilattice S has a bottom element $B := 0$ then $(S, +)$ is an idempotent commutative monoid with unit 0, i.e., $s + 0 = 0 + s = s$ for every $s \in S$. We denote this monoid as $(S, +, 0)$.

Conversely given an idempotent commutative monoid $S := (S, +, 0)$ with unit 0, one can define the semilattice (S, \leq) having the role

$$s \leq t \iff s + r = t \text{ for some } r \in S.$$

This role gives a proper poset. Indeed, $s \leq s$ (reflexivity) since $s + 0 = s$ for every s , and since $s_1 \leq s_2 \leq s_3$ iff there exist r_1, r_2 such that $s_1 + r_1 = s_2$ and $s_2 + r_2 = s_3$ which implies $s_1 + r_1 + r_2 = s_2 + r_2 = s_3$, thus $s_1 \leq s_3$ (transitivity). Finally (antisymmetry), $s \leq t \leq s$ iff there exist $r_1, r_2 \in S$ such that $s + r_1 = t$ and $t + r_2 = s$, but then

$$s + t = s + (s + r_1) = s + r_1 = t, \quad s + t = (t + r_2) + t = t + r_2 = s,$$

implying $s = t$.

Since S is a monoid, $s + t$ always exists, and thus we define

$$s \vee t = p := \min\{q \in S \mid q \geq s, t\}.$$

Moreover, since $p = s + x = t + y$ for some $x, y \in S$, then

$$p = p + p = s + t + x + y \geq s + t.$$

Having the above construction, we see that the category ICM of idempotent commutative monoids, whose maps are monoid homomorphism (i.e., $\phi : S \rightarrow S'$ such that $\phi(s+t) = \phi(s) + \phi(t)$ for any $s, t \in S$), is isomorphic to the category of SLAT whose objects are semilattices and its maps are sup-maps.

3.3. Lattices. We open again with a familiar definition.

Definition 3.9. A (finite) poset (P, \leq) in which each pair of elements p_i, p_j admits a **join** $p_i \vee p_j$ (also known as the least upper bound, or the supremum) and a **meet** $p_i \wedge p_j$ (also known as the greatest lower bound, or the infimum) is a (finite) **lattice**, written $L := (L, \leq)$.

Given a subset $X \subseteq L$, $X := \{x_1, \dots, x_m\}$, we write

$$\bigvee X := x_1 \vee x_2 \vee \dots \vee x_m, \quad \bigwedge X := x_1 \wedge x_2 \wedge \dots \wedge x_m,$$

respectively for the common join and meet of the members of X .

A poset (P, \leq) which has a join for each pair of elements and a (global) unique minimal element B , i.e., $B \leq p$ for all $p \in P$, called **bottom element**, is also a lattice, where the meet is defined by

$$p_i \wedge p_j := \bigvee \{q \in P \mid q \leq p_i, p_j\}, \quad \forall p_i, p_j \in P.$$

Note that $X := \{q \in P \mid q \leq p_i, p_j\}$ is nonempty since $B \in X$, formally we define $\bigvee \emptyset := B$. When a lattice L has a unique maximal element, we call this element the **top element** of L , and denote it T .

A lattice (L, \leq) is **distributive** if

$$s \wedge (t \vee t') = (s \wedge t) \vee (s \wedge t')$$

for all $s, t, t' \in L$. It is not difficult to show that this condition is equivalent to the dual condition

$$s \vee (t \wedge t') = (s \vee t) \wedge (s \vee t').$$

A lattice (L, \leq) is **complete** if for every subset $X \subseteq L$, the join $\bigvee X$ and the meet $\bigwedge X$ exist, where for $X = \emptyset$ we set

$$\bigvee \emptyset := B \quad \text{and} \quad \bigwedge \emptyset := T.$$

The **dual lattice** $L^* := (L^*, \leq)$ is defined over the same set of elements of L having the reversed order \leq , i.e., $L^* = L_{\text{rvs}}$ (cf. Definition 3.1).

A lattice map

$$\varphi : (L, \leq) \longrightarrow (L', \leq)$$

that satisfies $\varphi(\bigvee X) = \bigvee(\varphi(X))$ for all $X \subseteq L$ is called as before **sup-map**. Taking $X = \emptyset$, this implies $\varphi(B) = B'$. Similarly, φ is called an **inf-map** if $\varphi(\bigwedge X) = \bigwedge(\varphi(X))$ for all $X \subseteq L$, where now $\bigwedge \emptyset = T$, and thus $\varphi(T) = T'$. A map which is both sup-map and inf-map is termed **sup-inf-map**.

We set LAT to be the category of lattices whose map are sup-maps, which is a full subcategory of the category SLAT of semilattices. Both have the full subcategories FSLAT a and FLAT of finite semilattices and finite lattices, respectively, whose maps are sup-maps as well.

The reader should be note that the sup-maps preserve the structure of lattices partially, stronger maps are to be considered latter, incorporating the inf-maps.

Given a lattice $L := (L, \leq)$, where $X \subseteq L$, let us recall some definitions from [15, 6.1.2, p430].

Definition 3.10. Let ℓ, m be elements of a lattice $L := (L, \leq)$:

- (a) $\ell \in L$ is **strictly join irreducible** (sji) if whenever $\ell = \bigvee X$ there exists $x \in X$ such that $\ell = x$.
- (b) $\ell \in L$ is **join irreducible** (ji) if whenever $\ell \leq \bigvee X$ there exists $x \in X$ such that $\ell \leq x$.
- (c) $m \in L$ is called **strictly meet irreducible** (smi) if $m = \bigwedge X$ implies that there exists $x \in X$ such that $m = x$.
- (d) $m \in L$ is called **meet irreducible** (mi) if $m \geq \bigwedge X$ implies that there exists $x \in X$ such that $m \geq x$.

Join irreducibles are also called **primes**, while meet irreducibles are called **co-primes**.

For a finite lattice $L := (L, \leq)$ it is easy to see that the **smi**'s that are not T (the top element) are the unique minimal sets of meet generators of L , by universal algebra [13]. The top element T is meet generated by the empty set.

We define $\#_{\text{smi} \neq T}(L)$ to be the number of **smi**'s not T , i.e.,

$$\#_{\text{smi} \neq T}(L) := |\{\ell \in L \mid \ell \text{ is smi} \neq T\}|,$$

and similarly define the number of **smi**'s

$$\#_{\text{smi}}(L) := |\{\ell \in L \mid \ell \text{ is smi}\}|.$$

Dually, the **sjj**'s not B (the bottom element) are the unique minimal subsets of join generators of $L := (L, \leq)$, and we define their number to be

$$\#_{\text{sjj} \neq B}(L) := |\{\ell \in L \mid \ell \text{ is sjj} \neq B\}|,$$

and let

$$\#_{\text{sjj}}(L) := |\{\ell \in L \mid \ell \text{ is sjj}\}|.$$

Example 3.11. Let X_{2n} be a set of $2n$ elements, and let Λ_{2n} be the semilattice whose elements are all subset $Y \subseteq X$ of cardinality $\geq n$ of X_{2n} , together with the empty set \emptyset , and the parietal order determined by inclusion. The join of Λ_{2n} is set union and the determined meet is set intersection, unless its cardinality is less than n , which in this case is made \emptyset .

It is easy to see that all **sjj**'s not \emptyset of Λ_{2n} are all the subsets of order n of X_{2n} and the **smi**'s are all the subsets having $2n - 1$ elements. Thus, we have the following

$$|\Lambda_{2n}| = \frac{2^{2n} - \binom{2n}{2}}{2} + \binom{2n}{n} + 1,$$

where

$$\#_{\text{sjj}}(\Lambda_{2n}) = \binom{2n}{n} \quad \text{and} \quad \#_{\text{smi}}(\Lambda_{2n}) = 2n.$$

Therefore $\#_{\text{sjj}}$ and $\#_{\text{smi}}$ can be differ exponentially.

As shown in Corollary 3.5, the c-rank does not changed under reversing the order. On the other hand, Example 3.11 shows that $\#_{\text{sjj}}$ and $\#_{\text{smi}}$ are changing significantly; as $\#_{\text{sjj}}$ and $\#_{\text{smi}}$ are dual, they interchange when reversing the order. Unlike the situation of **sjj** and **smi** for finite lattices, whose members can be differ, we will see that the members of **ji**'s and **mi**'s are always equal.

Lemma 3.12. Suppose $\ell_1, \dots, \ell_{j-1}, \ell_{j_1} \vee \ell_{j_2}, \ell_j, \dots, \ell_k$ are independent in the lattice (L, \leq) . Then, for $i = 1$ or $i = 2$, $\ell_1, \dots, \ell_{j-1}, \ell_{j_i}, \ell_j, \dots, \ell_m$ are independent.

Proof. Write $\ell_j := \ell_{j_1} \vee \ell_{j_2}$ and let $U := \{m_1, \dots, m_k\}$ be the witness of $W := \{\ell_1, \dots, \ell_k\}$ in $A^c := A(L)^c$. Reordering the rows of A^c , we may assume that the witness $A^c[U, W]$ is of the form (1.2). Thus

$$m_s \not\leq \ell_t, \quad \text{for every } 1 \leq s < t \leq k,$$

and therefore $\ell_{j_1} \vee \ell_{j_2} = \ell_j \not\leq m_j$. So, either $\ell_{j_1} \not\leq m_j$ or $\ell_{j_2} \not\leq m_j$, say $\ell_{j_1} \not\leq m_j$. But then $U = \{m_1, \dots, m_k\}$ is also a witness for $W_1 := \{\ell_1, \dots, \ell_{j-1}, \ell_{j_1}, \ell_{j+1}, \dots, \ell_k\}$ being independent. \square

Proposition 3.13. $\text{c-rk}(L) \leq \#_{\text{sjj}}(L)$ for any finite lattice $L := (L, \leq)$.

Proof. Assume $m := \text{c-rk}(L)$, and let ℓ_1, \dots, ℓ_m be independent. Since the **sjj**'s not B are join generate L , each ℓ_j can be written as $\ell_j = \bigvee_{k=1}^m \ell_{j_k}$ where ℓ_{j_k} are **sjj**'s not B . Applying Lemma 3.12 inductively, we obtain an independent subset ℓ'_1, \dots, ℓ'_m where each ℓ'_j is **sjj** $\neq B$. (This is a stronger statement than Proposition 3.13.) \square

Corollary 3.14. $L := (L, \leq)$ has an independent subset X of maximal cardinality, .i.e., $|X| = \text{c-rk}(L)$, which is contained in $\{\ell \in L \mid \ell \text{ is sjj} \neq B\}$.

3.4. Spec of finite lattices. The previous section leads us to a spectral theory of finite lattices, see [15, §6-§7]. Following Marshal, Stone, and others, the basic approach of spectral theory of finite lattices is to consider the maximal distributive lattice generated by the set of surmorphisms. For this purpose, we need more structure.

Let (\mathbb{B}, \leq) , $\mathbb{B} := \{0, 1\}$, be the two element lattice with the standard partial order. Given an element $\ell \in L$, $L := (L, \leq)$ a lattice, we define the lattice map

$$\check{\varphi}_\ell : (L, \leq) \rightarrow (\mathbb{B}, \leq), \quad \check{\varphi}_\ell : x \mapsto \begin{cases} 1 & x \leq \ell, \\ 0 & \text{else.} \end{cases} \quad (3.2)$$

The map $\check{\varphi}_\ell$ is a sup-inf-map onto (\mathbb{B}, \leq) iff ℓ is **mi** (cf. Definition 3.10), and $\ell \neq T$. Conversely, a map $\varphi : (L, \leq) \rightarrow (\mathbb{B}, \leq)$ is sup-inf-map onto \mathbb{B} iff $\bigvee \varphi^{-1}(0) = \ell$ with $\ell \neq T$ an **mi** and $\varphi = \check{\varphi}$.

The dual result also holds, namely the map

$$\hat{\varphi}_\ell : (L, \leq) \rightarrow (\mathbb{B}, \leq), \quad \hat{\varphi}_\ell : x \mapsto \begin{cases} 1 & x \geq \ell, \\ 0 & \text{else.} \end{cases} \quad (3.3)$$

is a sup-inf-map onto (\mathbb{B}, \leq) iff ℓ is **ji**, and $\ell \neq B$. Conversely, a map $\varphi : (L, \leq) \rightarrow (\mathbb{B}, \leq)$ is sup-inf-map onto \mathbb{B} iff $\bigwedge \varphi^{-1}(1) = \ell$ with $\ell \neq B$ a **ji** and $\varphi = \hat{\varphi}$.

Given such a map $\varphi : (L, \leq) \rightarrow (\mathbb{B}, \leq)$ as above, we have a 1:1 correspondence

$$\bigvee \varphi^{-1}(0) \leftrightarrow \bigwedge \varphi^{-1}(1)$$

between **mi**'s not T and **ji**'s not B . Thus the number of **mi**'s not T equals that of **ji**'s not B , we denote this number $\#_{\text{mi} \neq T}$ and define

$$\mathfrak{B}(L) := \#_{\text{mi} \neq T}(L) - 1 = |\{\ell \in L \mid \ell \text{ is mi} \neq T\}| - 1. \quad (3.4)$$

We consider the spec lattice morphism for a finite lattice (L, \leq) :

$$\text{spec}(L) : (L, \leq) \rightarrow (\mathbb{B}, \leq)^{\mathfrak{B}(L)} \quad (3.5)$$

with

$$\text{spec}(L) = \Delta \left(\bigotimes_{\ell \text{ is mi}} \check{\varphi}_\ell \right) = \Delta \left(\bigotimes_{\ell \text{ is ji}} \hat{\varphi}_\ell \right)$$

where Δ is the diagonal map. (This equality derived by the above discussion.)

The map spec is a sup-inf lattice morphism of (L, \leq) onto the finite distributive lattice $(\mathbb{B}, \leq)^{\mathfrak{B}(L)}$, which is isomorphic to the all subsets of a set of cardinality $\mathfrak{B}(L)$ under inclusion. Since subsets of a distributive lattice are closed under meet and join containing B and T , then clearly they are distributive as well.

The image $\text{spec}(L)$ of spec is a distributive lattice, and we aim to show that it is the maximum distributive image of a sup-inf map of (L, \leq) . Using the notation of Definition 3.2, we deduce directly from the definition of spec that for any $\ell, \ell' \in L$

$$\text{spec}(\ell) = \text{spec}(\ell') \Leftrightarrow \ell^\uparrow \cap \{\text{mi} \neq T\} = \ell'^\uparrow \cap \{\text{mi} \neq T\} \Leftrightarrow \ell^\downarrow \cap \{\text{ji} \neq B\} = \ell'^\downarrow \cap \{\text{ji} \neq B\}.$$

Another important categorical notion is the Adjoint concept, known also as ‘‘Galois connection’’, for finite lattices which is as follows:

Proposition 3.15 (Adjoint of sup-maps). *Let (L, \leq) and (L', \leq) be two finite lattices. Assume that $\varphi : (L, \leq) \rightarrow (L', \leq)$ is a sup-map, and let $\psi : (L', \leq) \rightarrow (L, \leq)$ be the map (denoted also as $\varphi_{\text{adj-sup}}$) defined by*

$$\psi(\ell') := \varphi_{\text{adj-sup}}(\ell') := \bigvee \varphi^{-1}(\ell') \quad (3.6)$$

for each $\ell' \in L'$. Then, the following properties hold:

- (1) ψ is an inf-map,
- (2) $\varphi(\ell) \leq \ell' \Leftrightarrow \ell \leq \psi(\ell')$,
- (3) $\psi \circ \varphi \circ \psi = \psi$ and $\varphi \circ \psi \circ \varphi = \varphi$,
- (4) ψ is injective iff φ is surjective, φ is injective iff ψ is surjective,
- (5) $\varphi(\ell) = \bigwedge \psi^{-1}(\ell)$.

Proof. The proof is straightforward. \square

In general, for a finite lattice, **mi** implies **smi**, but the converse holds only for a finite distributive lattice.

Proposition 3.16. *The following are equivalent for a finite lattice (L, \leq) :*

- (1) (L, \leq) is distributive,
- (2) $mi \Leftrightarrow smi$ (resp. $ji \Leftrightarrow sji$),
- (3) **spec** is 1:1,
- (4) **spec** is a lattice isomorphism of (L, \leq) and $\text{spec}(L)$, with the induced order of $(\mathbb{B}, \leq)^{\mathbb{B}(L)}$.

Proof. (1) \Rightarrow (2): If p is **smi** and $p \geq a \wedge b$ then, by distributivity,

$$p = p \vee p = (a \wedge b) \vee p = (a \vee p) \wedge (b \vee p).$$

Then by **smi** $p = a \vee p$, say on $p \geq a$, and thus p is **mi**.

(2) \Rightarrow (3): The **smi**'s meet generate, so the **mi**'s ℓ meet generate L . Thus, if $\ell_1, \ell_2 \in L$, where $\ell_1 \neq \ell_2$, then there exists an **mi** m such that $\ell_1 \leq m$ and $\ell_2 \not\leq m$ (since each ℓ is the meet of all **mi**'s $\geq \ell$). Thus, m is different on ℓ_1 and ℓ_2 , and hence **spec** is 1:1.

(3) \Rightarrow (4): Set $\varphi := \text{spec}$. Since φ is a bijective sup-map, by Proposition 3.15, the adjoint map $\psi : \text{spec}(L) \rightarrow L$ exists and it is a bijective inf-map. To complete this part we need to show that $\varphi(\ell_1) \leq \varphi(\ell_2)$ implies $\ell_1 \leq \ell_2$. But, by Proposition 3.15, $\varphi(\ell_1) \leq \varphi(\ell_2)$ implies $(\psi \circ \varphi)(\ell_2) \geq \ell_2$ and since φ and ψ are bijections, $\ell_2 = (\psi \circ \varphi)(\ell_2)$, by definition of ψ .

(4) \Rightarrow (1) As already remarked, $\text{spec}(L)$ is a distributive lattice, and so is (L, \leq) by isomorphism. \square

Corollary 3.17 (Birkhoff). *(L, \leq) is a distributive lattice iff it is isomorphic to a collection of subsets of a finite set Z of cardinality $\mathbb{B}(L)$, closed under set theoretic union and intersection, including \emptyset and whole Z .*

Corollary 3.18. *$\text{spec}(L)$ is the unique maximal sup-inf image of $L := (L, \leq)$ which is a distributive lattice. $\text{spec}(L)$ is generated by $\mathbb{B}(L)$ elements, which is also the number of **smi**'s = **mi**'s not T (also is the number of **sji**'s = **ji**'s not B).*

Proof. $\text{spec}(L)$ is a distributive lattice, image of sup-inf map of (L, \leq) . To see the unique maximality, applying Proposition 3.16.(3) to any such image \tilde{L} ,

$$L \xrightarrow{\psi} \tilde{L} \xrightarrow[\text{iso}]{\text{spec}} \text{spec}(\tilde{L}),$$

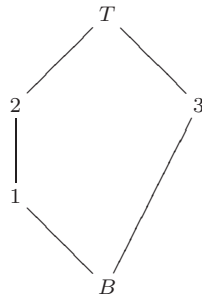
we see that $(\text{spec} \circ \psi)(L)$ factors through $L \rightarrow \text{spec}_L(L)$, and the rest of the proof follows from Proposition 3.16. \square

Remark 3.19. *If we endow $\{0, 1\}$ with the Sierpinski topology (not T_2) in which the closed sets are $\{\emptyset, \{0\}, \{0, 1\}\}$, then the null-kernel topology is induced topology of the poset topology. Thus, the closed sets of $\text{spec}(L)$ are of the form $V(\ell)$, for $\ell \in L$, with*

$$V(\ell) := \{m \in \text{spec}(L) \mid m \geq \ell\}.$$

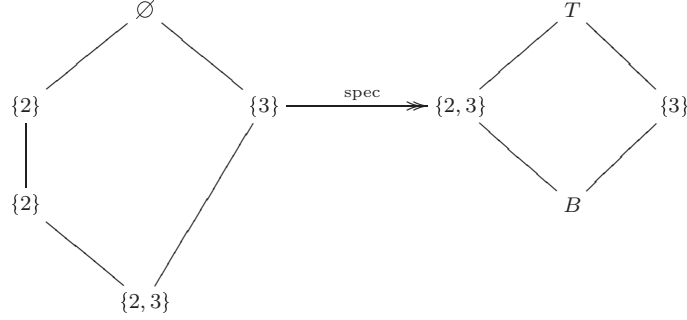
See [15, §7].

Example 3.20. *Let $L := (L, \leq)$ be the following lattice, $|L| = 5$,*



(3.7)

The mi 's not T are 2 and 3, so the relations on L given by spec are the singletons and $\{1, 2\}$.



Thus, for L we have $\#_{sj \neq B}(L) = \#_{smi \neq T}(L) = 2$, $\#_{ji \neq B}(L) = 3$, and $\mathfrak{B}(L) = 2$.

Example 3.21. Consider the semilattice Λ_{2n} in Example 3.11. Then, $\mathfrak{B}(\Lambda_{2n}) = 0$, $\text{spec}(\Lambda_n) = \{0\}$.

4. DIMENSION AND RELATED FUNCTIONS FOR FINITE LATTICES

In this section we develop the theory of c-rank of finite lattices, along with methods of computation and relations to other lattice functions studied earlier.

4.1. Duality of boolean modules. (See [15, Propositions 9.1.12-13].) Given a finite \mathbb{B} -module $M := M(\mathbb{B}, +)$, we define the dual module

$$M^* := \{\varphi : M \rightarrow \mathbb{B} \mid \varphi \text{ is a sup-map}\},$$

consisting of all sup-maps over M . Then, M^* itself is a \mathbb{B} -module closed under addition, i.e., satisfying $(f_1 + f_2)(m) = f_1(m) + f_2(m)$ for every $f_1, f_2 \in M^*$, whose unit is the constant zero mapping $f_0 : m \mapsto 0$ for every $m \in M$.

Remark 4.1. Clearly, for any $\ell \in L$, the sup-map $\check{\varphi}_\ell : (L, \leq) \rightarrow (\mathbb{B}, \leq)$, cf. Eq. (3.2), is contained in M^* and the map $\chi : \ell \mapsto \check{\varphi}_\ell$ is bijection, as is easy to prove.

Having this view, we deduce that if the \mathbb{B} -module M is considered as a lattice (M, \leq) , then the dual module M^* is the lattice $M_{\text{rvs}} := (M, \geq)$ obtained by reversing the order of M (realized as a lattice).

Similarly to the spec map (3.5), given for lattices, for a finite \mathbb{B} -module $L := (M, \leq)$, realized as a lattice, we define the map

$$\kappa_L := \Delta \left(\bigotimes_{\ell \in L} \check{\varphi}_\ell \right) : (L, \leq) \rightarrow \mathbb{B}^{|L|}, \quad (4.1)$$

a sup embedding of L into $\mathbb{B}^{|L|}$. So as in the spec case, L is order isomorphic to $\kappa(L)$ which is a subset of $\mathbb{B}^{|L|}$ closed under all joins. (Note that the meets of $\kappa(L)$ need not be the meets of $\mathbb{B}^{|L|}$.)

Given the boolean module $M := M(\mathbb{B}, +)$, and consider $\kappa(M) \subseteq \mathbb{B}^{|L|}$ realized as an $|L| \times |L|$ matrix C with coefficients in \mathbb{B} and whose rows are the members of $\kappa(M)$. Then, one could define $\text{c-rk}(M)$, the c-rank of M , to be the matrix rank $\text{rk}(C)$. This is the same as in Definition 2.1, since A^c equals the matrix C . Thus, the map $A \rightarrow A^c$ for the adjacency matrix $A := A(L)$ of (L, \leq) is given by passing from L to $\kappa(L)$ as in Remark 4.1. This is the same view as in the important [7].

4.2. Pullback and push of independent subsets. The following notion provides an important property of independence of subsets of lattices.

Definition 4.2. Given a lattice map $\varphi : (L, \leq) \rightarrow (L', \leq)$, we say that an element $\ell \in L$ is a **pullback** of $\ell' \in L'$ if $\varphi(\ell) = \ell'$.

Lemma 4.3. Let (L, \leq) and (L', \leq) be finite lattices and let $\varphi : (L, \leq) \rightarrow (L', \leq)$ be a sup-map of L to L' . Assume ℓ'_1, \dots, ℓ'_k are independent elements of L' and, taking one representative ℓ_i for each $\varphi^{-1}(\ell'_i)$, let $\ell_1, \dots, \ell_k \in L$ be their pullbacks. Then, ℓ_1, \dots, ℓ_k are independent in L .

Proof. Let $A := A(L)$ and $A' := A(L')$ be respectively the matrix structure of L and L' . Let $W' = \{\ell'_1, \dots, \ell'_k\} \subseteq L'$, and let $A^c[U', W']$ be a witness of W' for some $U' := \{m'_1, \dots, m'_k\}$. Assume φ is as in Proposition 3.15 and let

$$\psi := \varphi_{\text{adj-sup}} : (L', \leq) \rightarrow (L, \leq),$$

cf. Eq. (3.6). Define $m_i := \psi(m'_i)$ for each $i = 1, \dots, k$, and let $U := \{m_1, \dots, m_k\}$. We claim that $A^c[U, W]$ is a witness of $W := \{\ell_1, \dots, \ell_k\}$ in L . Indeed, permuting the columns of A^c , by Lemma 1.1, we may assume that $A^c[U, W]$ is of the Form (1.2).

If $\ell_j \leq m_j$, then applying φ – a sup-map – we get $\ell'_j \leq m'_j$ which is false. Thus, $\ell_j \not\leq m_j$, and we need to show that $\ell_i \leq m_j$ for $i < j$. But $\varphi(\ell_i) = \ell'_i \leq \varphi(m_j) = m'_j$, by Proposition 3.16. \square

Corollary 4.4.

- (i) If $\varphi : (L, \leq) \rightarrow (L', \leq)$ is an onto sup-map, then $\text{c-rk}(L) \geq \text{c-rk}(L')$,
- (ii) If (L', \leq) is a sub-module of (L, \leq) , i.e., a subset of L' closed under join, then $\text{c-rk}(L) \geq \text{c-rk}(L')$.

Proof. (i): Follows from Lemma 4.3.

(ii): Immediate by part (i). \square

We say that a finite \mathbb{B} -module M' **divides** a \mathbb{B} -module M , written $M' < M$, iff M' is the image of a sup-map of a sub-module of M . Accordingly, $M' < M$ implies $\text{c-rk}(M') < \text{c-rk}(M)$.

Theorem 4.5. For any finite lattice $L = (L, \leq)$ we have the equality $\text{c-rk}(L) = \text{ht}(L)$.

Proof. If $0 < \ell_1 < \dots < \ell_k$ is a chain in L , then ℓ_1, \dots, ℓ_k are independent with witness $U := \{m_1, \dots, m_k\}$, $m_1 = B$, $m_2 = \ell_1, \dots, m_k = \ell_{k-1}$. Thus, $\text{c-rk}(L) \leq k \leq \text{ht}(L)$.

Suppose $W := \{\ell_1, \dots, \ell_m\}$ are independent with a witness $A^c[U, W]$, $U := \{m_1, \dots, m_k\}$, of the Form (1.2). Then the chain

$$m_1 \wedge m_2 \wedge \dots \wedge m_k \leq m_2 \wedge m_3 \wedge \dots \wedge m_k \leq \dots \leq m_{k-1} \wedge m_k \leq m_k \leq T \quad (4.2)$$

is a strict chain in L , since $\ell_1, \dots, \ell_{k-1} \leq m_k$, $\ell_k \not\leq m_k$ then

$$\begin{array}{lll} \ell_1, \dots, \ell_{k-2} & \leq & m_{k-1} \wedge m_k, & \ell_{k-1}, \ell_k & \not\leq & m_{k-1} \wedge m_k, \\ \ell_1, \dots, \ell_{k-3} & \leq & m_{k-2} \wedge m_{k-1} \wedge m_k, & \ell_{k-2}, \ell_{k-1}, \ell_k & \not\leq & m_{k-1} \wedge m_k, \\ \vdots & & \vdots & \vdots & & \vdots \\ \ell_1 & \leq & m_2 \wedge \dots \wedge m_{k-1} \wedge m_k, & \ell_2, \dots, \ell_{k-1}, \ell_k & \not\leq & m_2 \wedge \dots \wedge m_{k-1} \wedge m_k. \end{array} \quad (4.3)$$

and $\ell_1 \not\leq m_1 \wedge \dots \wedge m_k$. Thus, $\text{c-rk}(L) \leq k \leq \text{ht}(L)$. \square

Remark 4.6. By Theorem 4.5, we see that $\mathbb{B}^{(n)} = \mathbb{B} \oplus \dots \oplus \mathbb{B}$ has rank n

Theorem 4.2 shows how to compute the c-rank of a given lattice, but we also want a way to compute independent subsets. To do so we need the following notion:

Definition 4.7. Let $L := (L, \leq)$ be a finite lattice, and let $U := \{m_1, \dots, m_k\}$ be a witness of $W := \{\ell_1, \dots, \ell_k\}$. A subset $\tilde{W} := \{\tilde{\ell}_1, \dots, \tilde{\ell}_k\} \subseteq L$ with $\tilde{\ell}_i \leq \ell_i$ and $\tilde{\ell}_i \not\leq m_i$ for every $i = 1, \dots, k$ is called a **push** of W with respect to U .

Proposition 4.8 (“Pushing”). A push of an independent subset W with witness U is independent with the same witness.

Proof. Clear, since U is a witness of \tilde{W} as well. \square

Proposition 4.9. Let $L := (L, \leq)$ be a finite lattice. The independent subsets of L are exactly pushes of chains of L . That is, if $W := \{\ell_1, \dots, \ell_k\}$ are independent with witness $U := \{m_1, \dots, m_k\}$, then, is an the proof Theorem 4.5,

$$\tilde{m}_1 < \tilde{m}_2 < \dots < \tilde{m}_k < T, \quad \tilde{m}_j = m_j \wedge \dots \wedge m_k,$$

is a strict chain. So

$$\tilde{\ell}_2 < \tilde{\ell}_3 < \dots < \tilde{\ell}_k < T,$$

is an independent set with witness $\tilde{m}_1, \dots, \tilde{m}_k$, and ℓ_1, \dots, ℓ_k is a push of this chain.

Proof. Clear by construction. \square

4.3. Lattice completion of finite posets. Given a poset $P := (P, \leq)$, let

$$P^\downarrow := \{p^\downarrow \mid p \in P\}, \quad (4.4)$$

where p^\downarrow is the down-set of p , cf. Definition 3.2. Then,

$$\text{Hs}(P) := (P^\downarrow, \subseteq) \quad (4.5)$$

is the **Hasse diagram** of P , a poset is by itself, whose partial order is determined by inclusion.

We define $\overline{P^\downarrow}$ to be the closure intersection of all subsets of P^\downarrow including the empty set. Clearly $\overline{P^\downarrow}$ contains the top element $T = P$. Then, $\overline{P^\downarrow}$ is the **Dedekind-MacNeille completion** of P , denoted also as $\text{DM}(P)$, and it is a finite complete lattice with meet set intersection, top element P , and determined join. Moreover P is order embedded into $\text{DM}(P)$ by

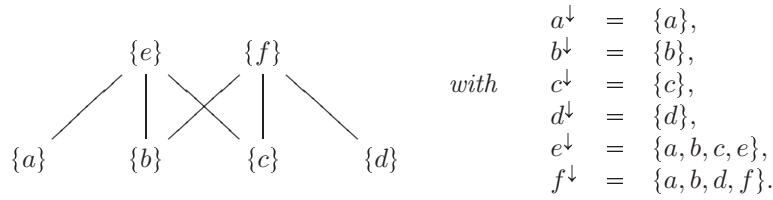
$$\Gamma : P \rightarrow \text{DM}(P), \quad \Gamma : p \mapsto p^\downarrow.$$

Similarly, we close all the subsets of P^\downarrow under union (including the empty set), and denote this **union closure** as $\text{UC}(P)$ – a finite complete lattice. This is a lattice completion of the poset P . The order ideals of P with joint set union, bottom element \emptyset and determined meet (which is just set intersection) shows that $\text{UC}(P)$ is a ring set, where $\Phi : p \mapsto p^\downarrow$ is an order embedding $\Phi : P \hookrightarrow \text{UC}(P)$, see [2, 3].

In some reasonable precise sense $\text{DM}(P)$ is the smallest lattice completion of the poset P , and $\text{UC}(P)$ is the largest lattice completion of P .

Remark 4.10. The Dedekind-MacNeille completion of a finite lattice $L := (L, \leq)$ is a lattice isomorphic to L (see [2, 3]).

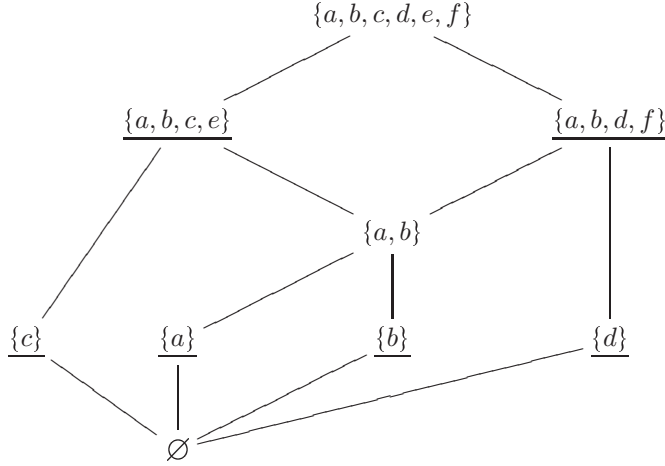
Example 4.11. Let $P := (P, \leq)$ be the 6-element poset $P := \{a, b, c, d, e, f\}$ whose Hasse diagram is



Computing the matrix $A := A(P)$, providing A^c , we get

$$A = \begin{array}{c|cccccc} \leq & a & b & c & d & e & f \\ \hline a & 1 & 0 & 0 & 0 & 1 & 0 \\ b & 0 & 1 & 0 & 0 & 1 & 1 \\ c & 0 & 0 & 1 & 0 & 1 & 1 \\ d & 0 & 0 & 0 & 1 & 0 & 1 \\ e & 0 & 0 & 0 & 0 & 1 & 0 \\ f & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \Rightarrow A^c = \begin{array}{c|cccccc} \not\leq & a & b & c & d & e & f \\ \hline a & 0 & 1 & 1 & 1 & 0 & 1 \\ b & 1 & 0 & 1 & 1 & 0 & 0 \\ c & 1 & 1 & 0 & 1 & 0 & 0 \\ d & 1 & 1 & 1 & 0 & 1 & 0 \\ e & 1 & 1 & 1 & 1 & 0 & 1 \\ f & 1 & 1 & 1 & 1 & 1 & 0 \end{array}$$

which shows that $\text{c-rk}(P) = \text{rk}(A^c) = 4$. The Dedekind-MacNeille completion $\text{DM}(P)$ of P is then



which is a lattice of height 4, and is an order embedding $P \hookrightarrow \text{DM}(P)$ of P into $\text{DM}(P)$. (The image of P in $\text{DM}(P)$ is indicated by underlines.)

Theorem 4.12. Let $\text{DM}(P)$ be the Dedekind-MacNeille completion of a poset $P := (P, \leq)$, with the order embedding $\Gamma : P \hookrightarrow \text{DM}(P)$. Abusing notation, we assume that Γ is the identity map, i.e., realized as $P \subseteq \text{DM}(P)$.

- (i) $\text{c-rk}(P) = \text{c-rk}(\text{DM}(P)) = \text{ht}(P)$.
- (ii) The independent subsets of P are those of $\text{DM}(P)$ restricted to P .

Proof. The proof follows the arguments of the proof of Theorem 4.5.

(i): Suppose $\text{ht}(\text{DM}(P)) = k$ and let

$$\emptyset = P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_{k-1} \subsetneq P_k = P$$

be a maximal chain of P . Define

$$Q_i := \bigcap \{p^\downarrow \mid p \in P_i\}, \quad i = 1, \dots, k,$$

to get a maximal chain

$$B = Q_k \subsetneq Q_{k-1} \subsetneq \cdots \subsetneq Q_1 \subsetneq Q_0 = T$$

in $\text{DM}(P)$. For every $i = 1, \dots, k$, pick

$$q_i \in Q_i \setminus Q_{i+1}.$$

Pick $s_1 \in Q_1$ so that $s_1 \geq q_1$ and $s_1 \not\geq q_0$, repeat this process recursively, picking $s_i \in Q_i$ such that $s_i \geq q_i$ and $s_i \not\geq q_{i-1}$. Accordingly, for each $i = 1, \dots, n$ we have

$$s_i \geq q_i, \dots, q_k \quad \text{and} \quad s_i \not\geq q_{i-1}.$$

This means that s_1, \dots, s_k provide a witness for the bottom element, i.e., it is of c-rank k .

(ii): Returning to the proof of Proposition 4.9, where if ℓ_1, \dots, ℓ_k in the lattice $\text{DM}(P)$ are independent with witness m_1, \dots, m_k , then setting

$$\tilde{m}_j = m_j \wedge \cdots \wedge m_k$$

so that

$$\tilde{m}_1 < \cdots < \tilde{m}_k < T$$

are an independent subset with witness $\tilde{m}_1, \dots, \tilde{m}_k$, and ℓ_1, \dots, ℓ_k is a push of this chain.

Then, similar to the proof of part (i), when $q_1, \dots, q_k \in P$ are points of P , we can find $\{s_1, \dots, s_k\} \subseteq P$ that is a witness for $\{q_1, \dots, q_k\} \subseteq P$. \square

4.4. Reformulation of “pushing-chains” to obtain independent subsets of finite posets. Let $\text{Hs}(L) := (L^\downarrow, \subseteq)$ be the Hasse diagram of a finite lattice $L := (L, \leq)$, cf. (4.5). Assign to each edge $(p_i^\downarrow, p_{i-1}^\downarrow)$ of $\text{Hs}(L)$, recording the relation $p_i^\downarrow \subset p_{i-1}^\downarrow$, the set theoretic difference

$$Q_i := p_{i-1}^\downarrow \setminus p_i^\downarrow.$$

Then, given a strict maximal chain

$$T^\downarrow = p_0^\downarrow > p_1^\downarrow > \cdots > p_{k-1}^\downarrow > p_k^\downarrow = B$$

of L from top to bottom in L , these Q_i are disjoint and their union equals $L \setminus \{B\}$.

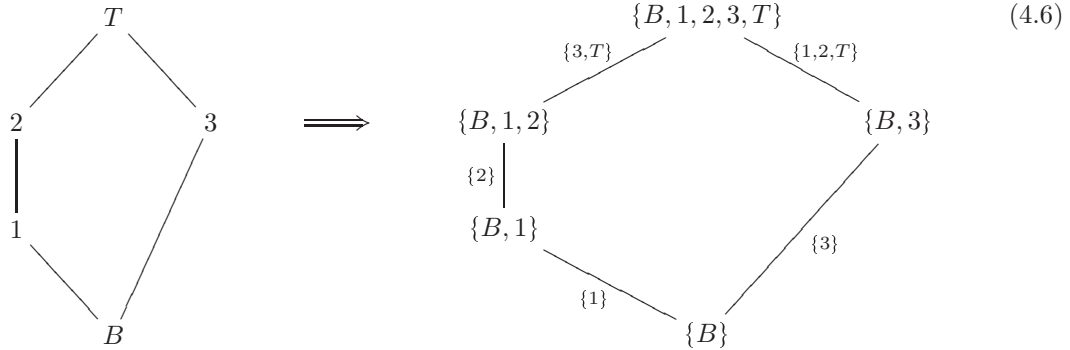
We call the collection

$$\mathcal{Q} := Q_1, \dots, Q_k$$

a **partition** of L . Note that these partitions correspond to different chains of L and thus could have different lengths.

Definition 4.13. A subset $X \subseteq L$ is a **partial cross section** of a partition \mathcal{Q} iff each $x \in X$ lies in a distinct Q_i , i.e., $|X \cap Q_i| \leq 1$ for each $i = 1, \dots, k$. (In such a case, we also say that X is an independent subset of \mathcal{Q} .) A **basis** of a partition \mathcal{Q} is a partial cross section X of maximal cardinality.

Example 4.14. Let (L, \leq) be the finite lattice (3.7) as in Example 3.20. Then, computing the Hasse diagram $\text{Hs}(L)$ and the differences along edges of maximal chains, we get



where each chain determines a partition of $L \setminus \{B\}$, i.e.

$$\{\{1\}, \{2\}, \{3, T\}\} \quad \text{and} \quad \{\{3\}, \{1, 2, T\}\}.$$

The bases of these partitions are therefore:

$$\{1, 2, 3\}, \quad \{1, 2, T\}, \quad \{3, T\},$$

(clearly are not of the same cardinality).

Theorem 4.15. A subset X of a lattice L is independent iff X is a partial cross section for the partition of L corresponding to some set theoretic maximal chain of L from top to bottom.

Proof. The proof is just a reformulation of “pushing chains” argument for lattices, Propositions 4.8 and 4.9. \square

(Important) Remark 4.16. The same construction defined above works for finite posets $P := (P, \leq)$ by considering the independent partitions of the Dedekind-MacNeille completion $\text{DM}(P)$. Then, restricting these partitions to P and taking the partial cross sections give the independent subsets of P . The proof is the same as before.

5. HEREDITARY COLLECTIONS

Recall Definitions 2.1 and 2.5, and the basic terminology, of [6]:

Definition 5.1. Let E be a set and let $\mathcal{H} \subseteq \text{Pw}(E)$ be a nonempty collection of subsets J of E . The nonempty collection \mathcal{H} is called **hereditary** if every subset J' of any $J \in \mathcal{H}$ is also in \mathcal{H} , more precisely:

- HT1: \mathcal{H} is nonempty,
 HT2: $J' \subseteq J, J \in \mathcal{H} \Rightarrow J' \in \mathcal{H}$.

(Hence, the empty set \emptyset is also in \mathcal{H} .) The pair $\mathcal{H} := (E, \mathcal{H})$, with \mathcal{H} hereditary over E , is called a **hereditary collection**.

A subset $J \in \mathcal{H}$ is called **independent**; otherwise is said to be **dependent**. A minimal dependent subset (with respect to inclusion) of E is called a **circuit**. A single element $x \in E$ that forms a circuit of $\mathcal{H} := (E, \mathcal{H})$, or equivalently it belongs to no basis, is called a **loop**. Two elements x and y of E are said to be **parallel**, written $x \parallel y$, if the 2-set $\{x, y\}$ is a circuit of \mathcal{H} . A hereditary collection is called **simple** if it has no circuits consisting of 1 or 2 elements, i.e., has no loops and no parallel elements.

Definition 5.2. We say that $\mathcal{H} = (E, \mathcal{H})$ satisfies the **point replacement property** iff

- PR: For every $\{p\} \in \mathcal{H}$ and every nonempty subset $J \in \mathcal{H}$ there exists $x \in J$ such that $J - x + p \in \mathcal{H}$.

Given a hereditary collection $\mathcal{H} = (E, \mathcal{H})$ that satisfies PR, then \mathcal{H} is simple iff all of its subsets of 2 or less element are independent. The proof is the same as for the matroid case.

Theorem 5.3 ([6, Theorem 5.3]). A vector hereditary collection [6, Definition 4.3] determined by the columns of a boolean matrix satisfies the point replacement property.

Theorem 5.4. If a simple hereditary collection $\mathcal{H} := (E, \mathcal{H})$ has a boolean representation, then there exist partitions $\mathcal{Q}_1, \dots, \mathcal{Q}_\ell$ of E so that the members of \mathcal{H} are the partial cross sections.

The statement of Theorem 5.4 can be strengthened to Theorem 5.5, basing on the construction as described next.

Let $\mathcal{H} := (E, \mathcal{H})$ be a simple hereditary collection, and assume it has a boolean representation $A := A(\mathcal{H})$. Augment the rows of A by all possible rows having exactly one entry 0 and the others 1; call this matrix B . Then augment the enlarged matrix B again by adding the sups of all possible row subsets, and denote this new matrix by A' .

Define the “closed sets” $C := \text{cl}(A')$ of A' by taking the collection of row-subsets of A' whose members have a 0-entry in r , for each row r of A' . (Denote such a row as $r_j^{(0)}$.) Then $\text{cl}(A')$ is closed under all intersections (so it includes E and the empty set \emptyset) and is also given by closing $\text{cl}(A)$ under all intersections. Thus, C is a lattice with meet intersection and determined join being $\text{cl}(X \cup Y)$, where closure of Z , a subset of subset E , is the intersection of all members of C containing Z .

Theorem 5.5. Let A' be as constructed above for a simple hereditary collection $\mathcal{H} := (E, \mathcal{H})$.

- (a) The rows of A' from a lattice $L' := \text{Lat}(A')$, under sup and determined join, which is sup-generated by the rows of A' .
- (b) The independent subsets of A and A' are the same by Lemma 3.12.
- (c) The map given by $r_i \mapsto r_i^{(0)}$ is a reverse isomorphism of $\text{Lat}(A')$ and $\text{cl}(A')$.
- (d) The partial cross sections of the partitions of $\text{cl}(A')$ give exactly \mathcal{H} .

Proof. (a) and (c) are clear, while (b) is obtained by Lemma 3.12.

(d): Consider a set theoretic maximal chain

$$\emptyset = C_k < C_{k-1} < \dots < C_1 < C_0 = E$$

from \emptyset to E in $\text{cl}(A')$. Since each C_j corresponds a row $r_j^{(0)}$, replacing C_j by the corresponding row $r_j^{(0)}$ and reversing the order of the chain we obtain the following chain in $\text{Lat}(A')$:

$$[0 \cdots 0] = r_0^{(0)} < \cdots < r_j^{(0)} < \cdots < r_k^{(0)} = [1 \cdots 1], \quad k \leq n, \quad (5.1)$$

where $r_0^{(0)}$ is a row whose entries are all 0 and $r_k^{(0)}$ is a row whose entries are all 1.

We number the element of E as e_i , where $i = 1, 2, \dots, n$ and $n = |E|$. By induction we can assume that each $r_j^{(0)}$ has all of its 1-entries first on the left and then all 1-entries. Let each $r_j^{(0)}$ have its 1-entries up to i_j in E , and consider the partition

$$\mathcal{Q} := \{1, \dots, i_1\}, \{i_1 + 1, \dots, i_2\}, \dots, \{i_{k-1} + 1, \dots, i_k\}, \quad i_k = n,$$

of E . By Proposition 4.8, Proposition 4.9, and §4.4, we see that the partial cross section of \mathcal{Q} are just the pushes of the chain (5.1). This proves (d). \square

Proposition 5.6. *Not any hereditary collection that satisfies PR (even if it turns out to be isomorphic to its dual) has a boolean representation.*

Proof. For example consider the hereditary collection $\mathcal{H} := (E, \mathcal{H})$ with $E = \{1, 2, 3, 4, 5\}$ whose bases are

$$B_1 := \{1, 2, 3\}, \quad B_2 := \{1, 2, 4\}, \quad B_3 := \{2, 3, 5\}, \quad B_4 := \{1, 4, 5\}, \quad B_5 := \{3, 4, 5\}. \quad (5.2)$$

It is easy to check that \mathcal{H} satisfies PR and is isomorphic to its dual \mathcal{H}^* (cf. [6, Definition 2.15]) whose bases are the 3-subsets of E excluding the bases of \mathcal{H} .

Since B_1 is a basis, E has a partition $\mathcal{Q} = Q_1, Q_2, Q_3$ with $i \in Q_i$, with 4 and 5 belong to these subsets Q_i 's. Since the bases are as given in (5.2), we are “enforced” to have the partition

$$Q_1 := \{1, 5\}, \quad Q_2 := \{2\}, \quad Q_3 := \{3, 4\}. \quad (5.3)$$

But then, by Theorem 5.4, $\{2, 4, 5\}$ which is not a basis is also independent – a contradiction. This means that \mathcal{H} can not have a boolean representation, since it must then be given by partial cross section of the partition, in particular $\{1, 2, 3\}$ must also be given in this way for which only (5.3) works. \square

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